CH-9 Sequence and Series (9.3)

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Q23 →

$$P^2 = (ab)^n$$

GP → $\underset{\text{First term}}{a}, ar, ar^2, \ldots, \underset{b\ (n^{th}\ term)}{ar^{n-1}}$

$$RHS = (ab)^n$$
$$= (a\ ar^{n-1})^n$$
$$= (a^2 r^{n-1})^n$$
$$= a^{2n} r^{n^2-n}$$

$$LHS = P^2$$
$$= [\underbrace{(a)(ar)(ar^2)\cdots(ar^{n-1})}_{n \text{ terms}}]^2$$
$$= [a^n r^{1+2+\cdots+n-1}]^2$$
$$[a^n r^{\frac{n(n-1)}{2}}]^2$$
$$= a^{2n} r^{n^2-n}$$

$1+2+\cdots+n =$

$$\boxed{\sum_{r=0}^{n} r = \frac{n(n+1)}{2}}$$

LHS = RHS

Q24 →

GP → $\underbrace{\overset{a_1}{a}, \overset{a_2}{ar}, \ldots\ldots, \overset{a_n}{ar^{n-1}}}, \underbrace{\overset{a_{n+1}}{ar^n}, \overset{a_{n+2}}{ar^{n+1}}, \ldots\ldots, \overset{a_{2n}}{ar^{2n-1}}}$

Sum of first n terms ⇒ $= a + ar + \cdots ar^{n-1}$

$$= \frac{a(r^n-1)}{(r-1)}$$

Sum of of next 'n' terms $= ar^n + ar^{n+1} + \cdots\cdots + ar^{2n-1}$

$$r^n(a + ar + \cdots + ar^{n-1})$$
$$r^n\left(a\left(\frac{r^n-1}{r-1}\right)\right)$$

$A = ar^n$

$R = r$

'n'

$$S = \frac{A(R^n-1)}{(R-1)}$$

$$S = \frac{ar^n(r^n-1)}{(r-1)}$$

$$\boxed{\frac{\text{Sum of First n terms}}{\text{Sum of next n terms}} = \frac{1}{r^n}}$$

— x — x — x —

Q25 →

a, b, c & d are in GP

Let common ratio be 'r'

Q →

$b = ar$

$$LHS = (a^2+b^2+c^2)(b^2+c^2+d^2)$$
$$= (a^2+a^2r^2+a^2r^4)(a^2r^2+a^2r^4+a^2r^6)$$

a
$b = ar$
$c = ar^2$
$d = ar^3$

LHS $= (a^2+b^2+c^2)(b^2+c^2+d^2)$

$= (a^2 + a^2r^2 + a^2r^4)(a^2r^2 + a^2r^4 + a^2r^6)$

$\Rightarrow a^2(1+r^2+r^4)\, a^2r^2(1+r^2+r^4)$

$\Rightarrow a^4r^2(1+r^2+r^4)^2$

RHS $= (ab+bc+cd)^2$

$(a.ar + ar.ar^2 + ar^2.ar^3)^2$

$(a^2r + a^2r^3 + a^2r^5)^2$ Taking a^2r common

$a^4r^2(1+r^2+r^4)^2$

Q26. $3, \_, \_, 81$

First terms $a = 3$ — (1)

Since we are inserting 2 terms, 81 is 4^{th} term

$ar^3 = 81$ — (2)

Second term / First inserted term $= ar$

$= 3 \times 3 = 9$

3^{rd} / 2^{nd} inserted $= ar^2$

$= 3 \times 3^2 = 27$

from (1) & (2)

$r^3 = 27 \Rightarrow \boxed{r = 3}$

Q27.

GM b/w a & b $= \dfrac{a^{n+1} + b^{n+1}}{a^n + b^n}$, Find 'n'

$\sqrt{ab} = \dfrac{a^{n+1} + b^{n+1}}{a^n + b^n}$

$a^{\frac{1}{2}} b^{\frac{1}{2}} (a^n + b^n) = a^{n+1} + b^{n+1}$

$a^{n+\frac{1}{2}} b^{\frac{1}{2}} + a^{\frac{1}{2}} b^{n+\frac{1}{2}} = a^{n+1} + b^{n+1}$

$0 = a^{n+1} - a^{n+\frac{1}{2}} b^{\frac{1}{2}} + b^{n+1} - a^{\frac{1}{2}} b^{n+\frac{1}{2}}$

$0 = a^{n+\frac{1}{2}}(a^{\frac{1}{2}} - b^{\frac{1}{2}}) + b^{n+\frac{1}{2}}(b^{\frac{1}{2}} - a^{\frac{1}{2}})$

at $n = \frac{-1}{2}$

$\dfrac{a^{\frac{1}{2}} + b^{\frac{1}{2}}}{\frac{1}{a^{\frac{1}{2}}} + \frac{1}{b^{\frac{1}{2}}}} \Rightarrow \dfrac{\cancel{\sqrt{a} + \sqrt{b}}}{\left(\frac{\cancel{\sqrt{b} + \sqrt{a}}}{\sqrt{ab}}\right)} \Rightarrow \boxed{\sqrt{ab}}$

$$0 = \left(a^{\frac{1}{2}} - b^{\frac{1}{2}}\right)\left(a^{n+\frac{1}{2}} - b^{n+\frac{1}{2}}\right)$$

$\Downarrow$

$a^{\frac{1}{2}} - b^{\frac{1}{2}} = 0$ OR $a^{n+\frac{1}{2}} - b^{n+\frac{1}{2}} = 0$

$a = b$ but a & b are different

$\boxed{a^{n+\frac{1}{2}} = b^{n+\frac{1}{2}}}$ $\boxed{a^0 = b^0 = 1}$

Since a & b are different

power $= 0$

$n + \frac{1}{2} = 0$

$\boxed{n = -\frac{1}{2}}$

Q28 → Let a & b be our two numbers...

$$a + b = 6\sqrt{ab}$$

divide by b

$$\frac{a}{b} + 1 = 6\sqrt{\frac{a}{b}}$$

Let $\boxed{\sqrt{\frac{a}{b}} = z}$ — ①

$$z^2 + 1 = 6z$$

$$z^2 - 6z + 1 = 0$$

$$z = \frac{6 \pm \sqrt{36-4}}{2}$$

$$z = \frac{6 \pm \sqrt{32}}{2}$$

$$z = \frac{6 \pm 4\sqrt{2}}{2}$$

$$z = 3 \pm 2\sqrt{2}$$

from ①

$$\sqrt{\frac{a}{b}} = 3 \pm 2\sqrt{2}$$

LHS is +ve

$\frac{a}{b} = (3+2\sqrt{2})^2$ or $(3-2\sqrt{2})^2$

Rationalization

Show $\frac{a}{b} = \frac{3+2\sqrt{2}}{3-2\sqrt{2}} \times \frac{3+2\sqrt{2}}{3+2\sqrt{2}} = \frac{(3+2\sqrt{2})^2}{1}$

More examples

Homogeneous Equation

$x^2 - 5xy + 6y^2 = 0$

Divide by y^2 or (x^2)

$$\frac{x^2}{y^2} - 5\frac{x}{y} + 6 = 0$$

$$z^2 - 5z + 6 = 0$$

Find $z = \frac{x}{y}$

Substitutions

Ex 2 → $z + 5\sqrt{z} + 6 = 0$ OR $z^4 - 5z^2 + 6 = 0$

$\Downarrow$

$x^2 + 5x + 6 = 0$

& solve

Case 1 ~~Show~~

$\boxed{a > b}$

So

$\frac{a}{b} = \frac{3+2\sqrt{2}}{3-2\sqrt{2}}$ which is $(3+2\sqrt{2})^2$

Case ②

$a < b$

$\frac{a}{b} = \frac{3-2\sqrt{2}}{3+2\sqrt{2}}$ which is $= (3-2\sqrt{2})^2$

Q29 → Let a & b be our two positive numbers...

AM = A

$\frac{a+b}{2} = A$ — ①

GM = G

$\sqrt{ab} = G$ — ②

$AM = A$ $\quad\quad$ $GM = G$

$$\boxed{\frac{a+b}{2} = A} \quad (1) \qquad\qquad \boxed{\sqrt{ab} = G} \quad (2)$$

Prove numbers are $a, b = A \pm \sqrt{A^2 - G^2}$

From (1) $\quad a + b = 2A \qquad\qquad$ from (2) $\quad b = \frac{G^2}{a} \quad (2')$

from (2') $\quad a + \frac{G^2}{a} = 2A$

$$\frac{a^2 + G^2}{a} = 2A$$

$$\underset{\sim}{1}a^2 - \underset{\sim}{2A}\, a + \underset{\sim}{G^2} = 0$$

$$a^2 - 2aA + G^2 = 0$$

$$a = \frac{-(-2A) \pm \sqrt{(-2A)^2 - 4(1)(G^2)}}{2(1)}$$

$$a = \frac{2A \pm 2\sqrt{A^2 - G^2}}{2} \qquad \boxed{a = A \pm \sqrt{A^2 - G^2}} \quad (3)$$

from (1) $\quad \frac{a+b}{2} = A$

$$b = 2A - a$$

$a = A + \sqrt{A^2 - G^2}$ $\qquad\qquad$ $a = A - \sqrt{A^2 - G^2}$

$b = 2A - (A + \sqrt{A^2 - G^2})$ $\qquad$ $b = 2A - (A - \sqrt{A^2 - G^2})$

$b = A - \sqrt{A^2 - G^2}$ $\qquad\qquad$ $b = A + \sqrt{A^2 - G^2}$

So two numbers are $\Rightarrow A \pm \sqrt{A^2 - G^2}$

$$A \pm \sqrt{(A+G)(A-G)}$$

Hence proved...

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Ex $\quad AM = 5$

$GM = 4$

$$a, b = A \pm \sqrt{A^2 - G^2}$$

$$= 5 \pm \sqrt{25 - 16}$$

GM = 4

$$= 5 \pm \sqrt{25-16}$$

$$\Rightarrow 5 \pm 3$$

$$8, 2$$

Q32 → Let roots of quadratic equation be α & β

$AM = \boxed{\frac{\alpha+\beta}{2} = 8}$ — (1) $GM = \boxed{\sqrt{\alpha\beta} = 5}$ — (2) Longer method

OR solve quadrati

$$\alpha, \beta = A \pm \sqrt{A^2 - G^2}$$

$$= 8 \pm \sqrt{64-25}$$

$$= 8 \pm \sqrt{39}$$

$$8+\sqrt{39},\ 8-\sqrt{39}$$

Quadratic equation

$$\boxed{y = C(x-\alpha)(x-\beta)}$$

$$y = C(x-8-\sqrt{39})(x-8+\sqrt{39})$$

$$y = C\left((x-8)^2 - (\sqrt{39})^2\right)$$

$$y = C(x^2-16x+64-39)$$

$$\boxed{y = C(x^2-16x+25)}$$

M2

Sum of Roots

$$\alpha+\beta = 16$$

Product of Roots

$$\alpha\beta = 25$$

Equation →

$$y = C\left(x^2 - (\alpha+\beta)x + \alpha\beta\right)$$

$$\boxed{y = C(x^2-16x+25)}$$

Special Series

$1+2+3+\cdots n = \frac{n(n+1)}{2}$ — (1) ✓ (AP)

$1^2+2^2+3^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$ — (2)

$1^3+2^3+3^3+\cdots n^3 = \left[\frac{n(n+1)}{2}\right]^2$ — (3)

(1) $1^2 - 0^2 = 2(1)-1$ $S_n = 1+2+\cdots+n$

(1) (1)

$$\cancel{1^2} - 0^2 = 2(1) - 1$$
$$\cancel{2^2} - \cancel{1^2} = 2(2) - 1$$
$$\cancel{3^2} - \cancel{2^2} = 2(3) - 1$$
$$\vdots$$
$$\cancel{(n-1)^2} - \cancel{(n-2)^2} = 2(n-1) - 1$$

General subtraction b/w squares of two consecutive numbers →
$$n^2 - \cancel{(n-1)^2} = 2n - 1$$

$$S_n = 1 + 2 + \cdots + n$$

$$n^2 - (n-1)^2$$
$$= n^2 - \{n^2 - 2n + 1\}$$
$$\Rightarrow 2n - 1$$

$$n^2 - 0^2 = 2(1 + 2 + 3 + \cdots + (n-1) + n) - \{1 + 1 + \cdots 1\}$$

$$n^2 = 2S_n - n$$

$$n^2 + n = 2S_n$$

$$\boxed{S_n = \frac{n(n+1)}{2}}$$

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(2) $S_n = 1^2 + 2^2 + \cdots + n^2$

$$\boxed{\begin{aligned} &n^3 - (n-1)^3 \\ &= n^3 - (n^3 - 3n^2 + 3n - 1) \\ &= 3n^2 - 3n + 1 \end{aligned}}$$

$$\cancel{1^3} - 0^3 = 3(1)^2 - 3(1) + 1$$
$$\cancel{2^3} - \cancel{1^3} = 3(2)^2 - 3(2) + 1$$
$$\cancel{3^3} - \cancel{2^3} =$$
$$\vdots$$
$$\cancel{(n-1)^3} - \cancel{(n-2)^3} = 3(n-1)^2 - 3(n-1) + 1$$
$$n^3 - \cancel{(n-1)^3} = 3(n)^2 - 3(n) + 1$$

$$n^3 - 0^3 = 3(1^2 + 2^2 + \cdots n^2) - 3(1 + 2 + \cdots n-1 + n) + (1 + 1 + 1 + \cdots + 1)$$

$$n^3 = 3S_n - 3\frac{n(n+1)}{2} + n$$

$$n^3 + \frac{3}{2}(n^2 + n) - n = 3S_n$$

$$\frac{2n^3 + 3n^2 + 3n - 2n}{2} = 3S_n$$

$$S_n = \frac{2n^3 + 3n^2 + n}{6}$$

$$S_n = \frac{n(2n^2 + 3n + 1)}{6}$$

$$2n^2 + 3n + 1$$
$$2n^2 + 2n + n + 1$$
$$2n(n+1) + 1(n+1)$$
$$(n+1)(2n+1)$$

$$\boxed{S_n = \frac{n(n+1)(2n+1)}{6}}$$

— x — x — x — x — x — x —

1
1 2 1
1 3 3 1
1 4 6 4 1

(3) $S_n = 1^3 + 2^3 + \cdots\cdots + n^3$

$n^4 - (n-1)^4 \rightarrow (n^2-2n+1)(n^2-2n+1)$

$= \cancel{n^4} - (\cancel{n^4} - 4n^3 + 6n^2 - 4n + 1)$

$4n^3 - 6n^2 + 4n - 1$

$$\cancel{1^4} - 0^4 = 4(1)^3 - 6(1)^2 + 4(1) - 1$$
$$\cancel{2^4} - \cancel{1^4} = 4(2)^3 - 6(2)^2 + 4(2) - 1$$
$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$
$$n^4 - \cancel{(n-1)^4} = 4(n)^3 - 6(n)^2 + 4(n) - 1$$

$$n^4 - 0^4 = 4\Sigma r^3 - 6\Sigma r^2 + 4\Sigma r - \Sigma 1$$

$$n^4 - 0^4 = 4S_n - \cancel{6}\left(\frac{n(n+1)(2n+1)}{\cancel{6}}\right) + \cancel{4}^{2}\left(\frac{n(n+1)}{\cancel{2}}\right) \quad -(n)$$

$$n^4 + n + n(n+1)(2n+1) - 2n(n+1) = 4S_n$$

$$n\left(n^3 + 1 + (n+1)(2n+1) - 2(n+1)\right) = 4S_n$$

$$n\left(n^3 \cancel{+1} + 2n^2 + 3n \cancel{+1} - 2n \cancel{-2}\right) = 4S_n$$

$$n^2(n^2 + 2n + 1) = 4S_n$$

$$S_n = \left[\frac{n(n+1)}{2}\right]^2 \quad \text{Hence proved}$$

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Ex 19

Remember ⟶

$S_n = a_1 + a_2 + \cdots\cdots a_n$

(Difficult) ✓ (1) Find a_n in terms of 'n' (method of differences)

(Easy) (Use special series) (2) $S_n = \sum_{r=0}^{n} a_r$

Ex 19

$$S_n = 5 + \overbrace{11 + 19 + 29 + 41 \cdots\cdots + a_n}^{n-1 \text{ terms}}$$

$$S_n = 5 + 11 + 19 + 29 + 41 \cdots + a_n$$
$$S_n = \quad 5 + 11 + 19 + 29 + \cdots + a_{n-1} + a_n$$

$$0 = 5 + \underbrace{(6 + 8 + 10 + 12 \cdots (a_n - a_{n-1}))}_{\text{AP of } n-1 \text{ terms}} - a_n$$

$$\left(S_{AP} = \frac{n}{2}(2a + (n-1)d)\right)$$

$$0 = 5 + \left(\frac{n-1}{2}\right)\left(2(6) + (n-1-1)2\right) - a_n$$

$$a_n = 5 + \left(\frac{n-1}{2}\right)(2n+8)$$

$$a_n = 5 + (n-1)(n+4)$$

$$\boxed{a_n = n^2 + 3n + 1}$$

$$S_n = \sum_{k=1}^{n} a_k$$

$$S_n = \sum_{k=1}^{n} (k^2 + 3k + 1)$$

$$S_n = \sum_{k=1}^{n} k^2 + 3\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1$$

$$S_n = (1^2 + 2^2 + \cdots n^2) + 3(1 + 2 + \cdots + n) + (1 + 1 + \cdots 1)$$

$$= \frac{n(n+1)(2n+1)}{6} + 3\frac{n(n+1)}{2} + n$$

$$= n\left(\frac{2n^2+3n+1}{6} + \frac{3n+3}{2} + 1\right)$$

$$= n\left(\frac{2n^2+3n+1+9n+9+6}{6}\right)$$

$$= n\left(\frac{2n^2+12n+16}{6}\right) \Rightarrow \frac{n(n^2+6n+8)}{3}$$

$$= \boxed{\frac{n(n+2)(n+4)}{3}}$$

— x — x — + —

Ex 20 →

$$a_n = n(n+3)$$

$$S_n = \sum_{k=1}^{n} a_k$$

Series, ⇒ $a_1, a_2, a_3, a_4, a_5, \ldots$

4, 10, 18, 28, 40, ...

$$S_n = \sum_{k=1} a_k$$

$$S_n = \sum_{k=1}^{n} k(k+3)$$

$$S_n = \sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} 3k$$

$$S_n = \frac{n(n+1)(2n+1)}{6} + 3\frac{n(n+1)}{2}$$

$$S_n = \frac{n(n+1)}{2}\left\{ \frac{2n+1}{3} + 3 \right\}$$

$$S_n = \frac{n(n+1)(2n+10)}{2 \times 3}$$

$$\boxed{S_n = \frac{n(n+1)(n+5)}{3}}$$

4, 10, 18, 28, 40, - - -

$$S_n = 4 + 10 + 18 + 28 + 40 \cdots + a_n$$

$$S_n = \underbrace{4 + 10 + 18 + 28 \quad + a_{n-1}}_{n-1 \text{ terms}} + a_n$$

$$a_n = 4 + (6 + 8 + 10 + \cdots n\text{-}1 \text{ terms}) - a_n$$

$$a_n = 4 + 6 + 8 + \cdots n \text{ terms}$$

$$a_n = \frac{n}{2}\left(2(4) + (n-1)2\right)$$

$$a_n = n(4 + n - 1)$$

$$a_n = n(n+3)$$

— x — x — x — x

$n = 1$

$$\frac{1 \times 2 + 6}{3} = 4 \qquad a_1 = 4$$

$n = 2$

$$\frac{2(3)(7)}{3} = 14 \qquad a_1 = 4 \quad a_2 = 10$$